Unexpected robustness against noise of a class of nonhyperbolic chaotic attractors

Holger Kantz,[1] Celso Grebogi,[2] Awadhesh Prasad,[3] Ying-Cheng Lai,[3,4] and Erik Sinde[1]
[1]*Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Strasse 38, 01187 Dresden, Germany*
[2]*Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05315-970 São Paulo, Brazil*
[3]*Department of Mathematics, Arizona State University, Tempe, Arizona 85287*
[4]*Departments of Electrical Engineering and Physics, Center for Systems Sciences and Engineering, Arizona State University, Tempe, Arizona 85287*

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Chaotic attractors arising in physical systems are often nonhyperbolic. We compare two sources of nonhyperbolicity: (1) tangencies between stable and unstable manifolds, and (2) unstable dimension variability. We study the effects of noise on chaotic attractors with these nonhyperbolic behaviors by investigating the scaling laws for the Hausdorff distance between the noisy and the deterministic attractors. Whereas in the presence of tangencies, interactive noise yields attractor deformations, attractors with only dimension variability are robust, despite the fact that shadowing is grossly violated.

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I. INTRODUCTION

In the study and applications of the ergodic theory of chaotic attractors in deterministic dynamical systems, the effect of noise is an issue of paramount importance. In laboratory experiments and in numerical simulations of chaotic systems, the deterministic time evolution is rarely observed in the deterministic sense, because of the presence of inevitable environmental noise or computer round off and their interplay with the underlying dynamical process. Whereas computer round off yields a perturbed deterministic dynamics, experimental systems must be considered as *nonlinear stochastic processes*, with however small is the stochastic component in comparison with the other factors governing the time evolution of the system. The *dynamical* or *interactive* noise interferes with the deterministic part of the dynamics and, as many studies show [1,2], this type of noise can have considerable effects that are much more severe than just reducing the information about the current state of the systems as in the case of *measurement* or *observational noise* [3]. Because of the sensitivity on initial conditions exhibited by chaotic systems, which causes an exponential divergence of nearby trajectories, the correlation between the noisy trajectory and the deterministic one having the same initial conditions decay generally rapidly in time.

However, in hyperbolic chaotic systems, in which the phase space is locally spanned by a fixed number of *distinct* stable and unstable directions that are transported consistently under the dynamics, the effect of noise is not so severe in the following sense. While a noisy trajectory diverges exponentially from the true one with the same initial condition, typically there exists a true trajectory, with a slightly different initial condition, that stays close to the noisy one. That is, for a hyperbolic system, an experimentally or numerically obtained trajectory is typically shadowed, within a small distance in the phase space, by a true trajectory of the ideal physical system, a desirable fact that is mathematically guaranteed by the shadowing lemma due to Anosov and Bowen [4].

Nonetheless, chaotic attractors arising in physical situations are typically nonhyperbolic. Two distinct types of nonhyperbolicity are frequently observed: (1) tangencies between stable and unstable manifolds and, (2) unstable dimension variability. In the former case, there exists a dense set of points in the phase space at which the stable and unstable eigendirections are indistinguishable (tangencies). Since the shadowing of a noisy trajectory relies on the existence of distinct stable and unstable directions [4], tangencies between the stable and unstable manifolds render impossible indefinite shadowing. However, shadowing of noisy trajectories for a finite amount of time can still be expected insofar as the trajectory stays away from any tangency point. For low-dimensional chaotic systems (systems with only one unstable direction) that have quadratic tangencies, the shadowing time $T(\epsilon)$ scales algebraically with the noise amplitude ϵ as $T(\epsilon)\sim\epsilon^{-1/2}$ [1]. Thus, if the noise level is small, shadowing can still be expected for a reasonable amount of time. The problem of shadowing for systems with unstable dimension variability [5] can, however, be extremely severe [6,7]. For such a system, there is no continuous splitting of the stable and unstable directions along a trajectory. As a result, the shadowing times can be very short [7]. Despite this fundamental difficulty in shadowing, it was recently pointed out that for certain chaotic systems, apparently statistical quantities such as various dynamical invariants can still be obtained through physical or numerical experiments of such systems [8].

While the issue of shadowing is concerned with individual trajectories, it is often more relevant to study the dependence of the invariant measure and of the set in phase space supporting this measure on the noise level. The effect of noise on the global structures of nonhyperbolic attractors with tangencies has recently been investigated [2,9]. The aim of this paper is to address how noise affects chaotic attractors with either or both sources of nonhyperbolicity. In particular, in order to measure how a chaotic attractor is geometrically deformed by noise, we utilize the Hausdorff distance (to be defined in Sec. II) between attractors with and without noise [10], and investigate how the distance scales with the noise amplitude. Our principal and very surprising result is that, while noise can significantly influence the global structures of nonhyperbolic attractors with tangencies (the Hausdorff distance increases algebraically with the noise level with a large constant of proportionality), its effect on attractors with unstable dimension variability seems to be negligible in the sense that it is as weak as on hyperbolic attractors. We shall present general arguments and detailed numerical results for two systems with unstable dimension variability. In chaotic systems with both tangencies and unstable dimension variability, the attractor deformation due tangencies is the dominating effect.

The remainder of the paper is organized as follows. In Sec. II, we describe unstable dimension variability, define the Hausdorff distance, and discuss how it should scale for different types of nonhyperbolic attractors. In Sec. III, we construct a class of numerical systems to address the scaling issue. A discussion including reference to the mathematical literature is presented in Sec. IV.

II. UNSTABLE DIMENSION VARIABILITY AND NOISE SCALING OF THE HAUSDORFF DISTANCE

Consider an N-dimensional map: $\mathbf{x}_{n+1}=\mathbf{F}(\mathbf{x}_n,p)$, where $\mathbf{x}_n\in\mathcal{R}^N$ and p is a system parameter. We focus on parameter regimes in which the ergodic invariant set is a chaotic attractor. The set is hyperbolic if the following three conditions are met [11]:

(1) At each point in the set, the tangent space can be split into an expanding subspace and a contracting subspace. Distances in the expanding (contracting) subspace grow (shrink) exponentially in time on average.

(2) The angle between the stable and the unstable subspaces is bounded away from zero.

(3) The expanding subspace evolves into the expanding one along a typical trajectory and the same is true for the contracting subspace.

Violation of condition (2) leads to nonhyperbolicity with tangencies, which occurs commonly in low-dimensional chaotic systems with only one unstable direction. Nonhyperbolicity with unstable dimension variability is caused by the violation of condition (3), which can occur in chaotic systems with more than one unstable or more than one stable direction. In high dimensions, commonly there are systems that violate both conditions (2) and (3).

For nonhyperbolic systems with tangencies, the shadowing time can still be large in the presence of small noise, because of the algebraic scaling of the time [1]. For nonhyperbolic systems with unstable dimension variability, the shadowing time can be oppressively small [7]. This can be understood by considering a simple ergodic invariant set containing two unstable fixed points: one with a single local unstable direction and one with two local unstable directions [12]. Trajectories wandering in the invariant set can spend arbitrarily long times near each point. Imagine a ball of initial conditions starting near the fixed point with the single unstable direction. Under the dynamics, the complement of the unstable direction is contracting, so the ball of true trajectories will be squeezed into a very thin line along the unstable direction. Due to noise of amplitude ϵ, points on any experimental trajectory $\mathbf{x}_{\rm noise}$ are typically found at a distance ϵ away from all true trajectories. When the trajectory visits the neighborhood of the second fixed point with two unstable directions, the small distance between $\mathbf{x}_{\rm noise}$ and all true trajectories increases exponentially along the *new* unstable direction at a rate determined by the average expanding eigenvalue associated with this direction. Thus, whenever this happens, the noisy trajectory immediately diverges exponentially from the true ones. For a chaotic set with unstable dimension variability, this is by no means a rare phenomenon. In fact, each single set of unstable periodic points with given numbers of unstable directions is believed to be *dense* in the chaotic set, substantially reducing the time a noisy trajectory can be expected to remain close to any true trajectory of the system.

To characterize the effect of noise on nonhyperbolic chaotic attractors, we utilize the concept of Hausdorff distance that measures the distance between two sets. Let $\mathcal{X}$ denote the noisy attractor and $\mathcal{Y}$ be the noiseless attractor. The distance between a point $\mathbf{x}\in\mathcal{X}$ and the noiseless attractor is defined to be:

$$d_{\rm min}(\mathbf{x},\mathcal{Y})=\min\{\|\mathbf{x}-\mathbf{y}\|,\mathbf{y}\in\mathcal{Y}\},\qquad(1)$$

where $\|\cdot\cdot\|$ is the Euclidean distance between the points $\mathbf{x}$ and $\mathbf{y}$. We call a Hausdorff pseudodistance from set $\mathcal{X}$ to set $\mathcal{Y}$

$$d_{\rm Hausd}(\mathcal{X},\mathcal{Y})=\max\{d_{\rm min}(\mathcal{X},\mathcal{Y}),\mathbf{x}\in\mathcal{X}\}.\qquad(2)$$

The Hausdorff distance between the two sets is defined as

$$D=\max\{d_{\rm Hausd}(\mathcal{X},\mathcal{Y}),d_{\rm dausd}(\mathcal{X},\mathcal{Y})\}.$$

In the following, we will discuss sets $\mathcal{X}$ that are blown up versions of the sets $\mathcal{Y}$. Hence the distance D will be given by the value of $d_{\rm Hausd}(\mathcal{X},\mathcal{Y})$, since $d_{\rm Hausd}(\mathcal{Y},\mathcal{X})=0$ in these cases.

In order to be useful, this definition contains the implicit assumption that the largest value $d_{\rm min}$ is finite. In a situation involving Gaussian noise, this is not guaranteed. But even if $d_{\rm Hausd}$ is finite, the actual value of $d_{\rm Hausd}$ computed on a *finite* sample of the set $\mathcal{X}$ depends strongly on the realization of the sample, since it is solely determined by the sampling of the "tail" of the distribution of $d_{\rm min}$. Hence, in order to derive statistically robust numbers for finite samples, we define $\hat{d}_{\rm Hausd}$ by the smallest value such that not more than k points of the finite set $\mathcal{X}$ have a distance $d_{\rm min}$ larger than $\hat{d}_{\rm Hausd}$ (this is a biased but consistent estimator of the Hausdorff distance with a reasonably small variance for $k=10$).

For nonhyperbolic chaotic attractors with tangencies, a trajectory typically wanders in hyperbolic regions until a homoclinic tangency is encountered. If one considers an ensemble of noisy trajectories, they are systematically driven out of the neighborhood of the attractor *only* at the homoclinic tangencies. At a tangency, a prolongation of the structure of the attractor can be observed, which depends on the noise level and the local expansion rate near the tangency [2]. When the noisy trajectory is located in the hyperbolic region of the attractor, it is generally pushed back to the attractor. Quantitatively, the Hausdorff distance was proposed in Ref. [9] to scale with the noise amplitude as $d_{\mathrm{Hausd}}(\epsilon)\sim\epsilon^{1/D_1}$, where D_1 is the information dimension. In contradiction to this scaling law, Ref. [2] and the numerics presented below suggest the simpler scaling law

$$d_{\mathrm{Hausd}}(\epsilon)\approx c\,\epsilon, \tag{3}$$

which also holds trivially for hyperbolic systems. Compare also Sec. IV for a mathematical assertion for the general validity of Eq. (3).

The important finding of Ref. [2] is that the factor of proportionality c in Eq. (3) can be huge, as will also be confirmed in Sec. III. Since the main contribution to the Hausdorff distance comes from the primary tangency points (tangencies with the smallest curvature) and their immediate images, for a noisy trajectory of finite length T, the scaling is observable for $\epsilon\gtrsim\epsilon_c\sim 1/T^{\alpha}$, where $\alpha\approx 2$ if the tangency is quadratic. For two-dimensional maps, such as the Hénon map with negative Jacobian, the primary tangency points typically are located at the outmost boundary of the closure of the attractor.

For attractors with unstable dimension variability as the sole source of nonhyperbolicity, the shadowability of noisy trajectories is severely limited. One might hence expect to see also considerable differences between the attractors of noise-free systems and noise-driven systems. However, a closer investigation of the structure of the stable and unstable manifolds suggests that the global features of the attractor remain almost unaffected by noise. The heuristic arguments in support of this will be given below, and will be illustrated in detail using a dynamical system introduced in this paper.

For the explanation of the unexpected robustness against noise, we have to distinguish between the two cases in which the direction corresponding to the fluctuating Lyapunov exponent is either expanding on average or contracting on average. In the latter case, i.e., if the Lyapunov exponent, whose finite time average has fluctuating sign [6] (which is responsible for the unstable dimension variability), is negative, the periodic points with more than the average unstable directions are typically embedded deep inside in the closure of the attractor. The set of points forming the attractor boundary has no more unstable directions than there are positive Lyapunov exponents. If the converse were true, the boundary could not be a boundary of an attractor (it could only, e.g., be the boundary of a repeller).[1] To recall, the natural invariant measures on attractors are such that the expanding directions are filled in a continuous way, hence, all directions transverse to the attractor boundary have to be linearly attracting. As a consequence, low amplitude noise cannot take a trajectory away from the attractor. This argument also holds for attractors with homoclinic tangencies everywhere where linear stability analysis governs the growth of perturbations. Although nowhere any direction transverse to the attractor is linearly expanding, due to the degeneracy of the manifolds at tangency points, one tangent space direction is not governed by linear stability analysis at all. Nonlinear effects are then dominant already for vanishing noise levels and lead to noise amplification.

If the direction responsible for unstable dimension variability carries a positive Lyapunov exponent, it is a globally expanding direction. Hence, the projection of the invariant measure onto this direction is continuous and insensitive to noise.

In all cases, dynamical noise makes attractors slightly fuzzy, i.e., fractal structure and attractor boundaries are slightly blurred. This effect exists also for strictly hyperbolic systems and is comparable to measurement noise. It introduces a Hausdorff distance between the noise free and the noisy attractor that is proportional to the noise level as in Eq. (3), but with a factor of proportionality c, which is less than unity and that is determined by the weakest contracting direction that is not filled continuously (i.e., the one in the Kaplan Yorke formula that contributes as a fraction). This should be contrasted with a huge c value in the case of homoclinic tangencies. For nonhyperbolic attractors with both unstable dimension variability and tangencies, we thus expect the effect of tangencies to dominate in the sense that the Hausdorff distance obeys the scaling in Eq. (3) with a large constant of proportionality.

The finiteness of the sample representing the noise-free attractor introduces a minimal Hausdorff distance. This can be obtained numerically by the computation of the distance between two different noise-free samples of the same attractor. Only if the noise induced effects are larger than this finite sample effect, the scaling laws can be seen.

Hence, in the numerics presented below, two scaling regimes arise. At small noise levels, the Hausdorff distance is noise independent. At large noise levels, we expect to observe the scaling of Eq. (3) for the system with homoclinic tangencies. For hyperbolic systems or systems with unstable dimension variability, we expect also a crossover to a regime where the Hausdorff distance is proportional to the noise level, but only at much larger noise levels corresponding to small c.

III. MODEL SYSTEMS AND NUMERICAL RESULTS

A. Model systems with unstable dimension variability

As a first model system, we introduce a model for dimension variability, namely,

$$x_{n+1}=2x_n+3y_n \bmod(1),$$

$$y_{n+1}=3x_n+5y_n \bmod(1),$$

$$z_{n+1}=\frac{2}{\pi}\arctan\left\{\frac{\alpha\pi}{2}[z_n+\sin(2\pi x_n-2\pi y_n)]\right\}, \tag{4}$$

where $\mathbf{x}\equiv(x,y)$ obeys the hyperbolic two-dimensional cat map, and α is a parameter that can be adjusted to produce

[1]This argument does not apply if the chaotic attractor is confined to an invariant subspace. For such situations noise effects known as ‘‘attractor bubbling’’ [15] can occur, where noisy and deterministic attractors can be very different. However, symmetry and invariance of subspaces are structually unstable phenomena that we exclude in our discussion.

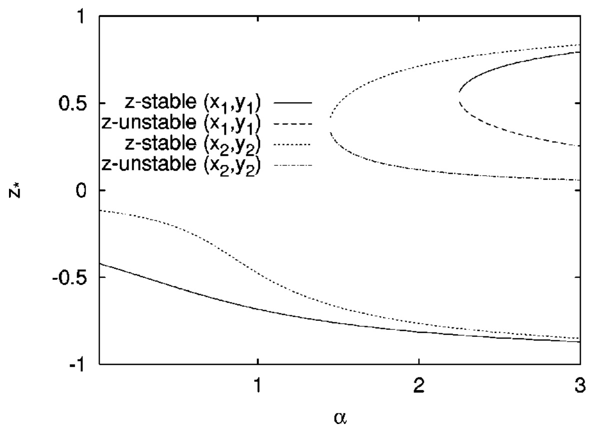

FIG. 1. Bifurcation diagram of z coordinates of Eq. (4) of the two fixed points with the (x,y) coordinates (2/5,1/5), (3/5,4/5). The z coordinates of the two other fixed points are obtained by $z \mapsto -z$.

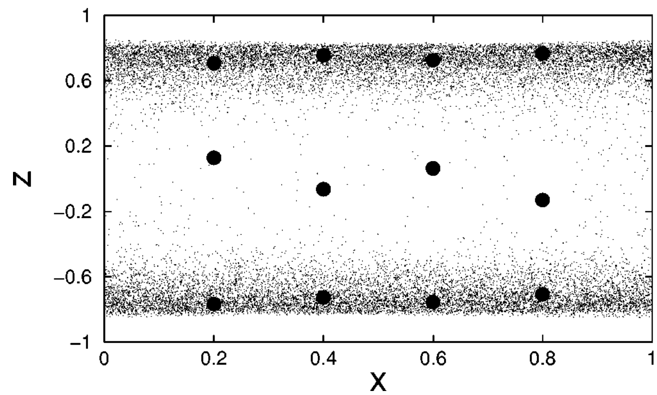

FIG. 2. Attractor (dots) of Eq. (4) projected into the (x-z) plane for $\alpha=2$. The fixed points are shown by filled circles (●).

unstable dimension variability. The cat map has five saddle fixed points: $\mathbf{x}_0=(0,0)$, $\mathbf{x}_1=(2/5,1/5)$, $\mathbf{x}_2=(3/5,4/5)$, $\mathbf{x}_3=(1/5,3/5)$, and $\mathbf{x}_4=(4/5,2/5)$ with eigenvalues of the Jacobians $(7\pm3\sqrt{5})/2$. Each of them gives rise to at least one and at most three fixed points of the full three-dimensional system. For $\alpha<1$, a single fixed point with two stable and one unstable direction is created from every saddle point of the cat map, as can be easily verified. At $\alpha=1$, the fixed point (0,0,0) undergoes a pitchfork bifurcation, where the point (0,0,0) gains a second unstable direction, and where two new fixed points $(0,0,z_0)$ and $(0,0,-z_0)$ are born, both with one unstable and two stable directions. At $\alpha\approx1.45$ and $\alpha\approx2.25$, also the first and the second, respectively, pair of the remaining four fixed points undergo subcritical tangent bifurcations, i.e., in addition to the existing singly unstable fixed point, a pair of one singly and one doubly unstable fixed point is formed (see Fig. 1). Hence, for $\alpha>2.25$, the full three-dimensional system, Eq. (4), has 15 fixed points. Their locations (x_*,y_*,z_*) in the phase space, where z_* is the solution of $z_*=2/\pi\arctan\{\alpha\pi/2[z_*+\sin(2\pi x_*-2\pi y_*)]\}$, depend on x_*, y_*, and α. The eigenvalues of the Jacobians of each fixed point are $(7\pm3\sqrt{5})/2$ and

$$\alpha\Bigg/\left(\left\{\frac{\alpha\pi}{2}[z_*+\sin(2\pi x_*-2\pi y_*)]\right\}^2+1\right).$$

These 12 fixed points, together with the projection of the chaotic attractor in the (x,z) plane, are shown in Fig. 2.

In a similar way, every period p orbit of the cat map gives rise to either a single period p orbit of the full system, or to a triple of period p orbits, one of which is doubly unstable, the two others being singly unstable. The larger the parameter α is, the larger is the number of orbits that have bifurcated. It seems that there are no other bifurcations than these, such that in the limit of very large α, the system has a triplet of orbits for every orbit of the cat map. The feature that is relevant for the understanding of the robustness of the attractor against noise is the following: for every periodic point (x,y) of the cat map, the points of the orbits of the full system with largest (z_+) and with smallest z component (z_-) belong to the singly unstable orbits, the z component of whose tangent space being stable. The z coordinate z_d of the doubly unstable orbit fulfills $z_-<z_d<z_+$. Moreover, a one-dimensional subspace of the unstable manifold of the doubly unstable orbit is identical with one direction of the stable manifold of the singly unstable orbits, and this homoclinic orbit is given by the corresponding line connecting the two types of orbits. This proves, for this particular system, how the second unstable direction can be such that on the average the corresponding Lyapunov exponent is negative, and how the attractor boundary can be in fact linearly attracting. Hence, no noise amplification mechanism outside the attractor is present. Internally, however, the doubly unstable periodic orbits form a kind of local separatrix, since initial conditions with identical (x,y) coordinates converge to either the z_+ or the z_- orbit, depending on whether the initial z value was larger or smaller than the corresponding z_d. This is relevant for the understanding of the lack of shadowing in this map, which will be discussed elsewhere in detail. The general lack of shadowability of systems with dimension variability was studied in Ref. [7].

Another, physically motivated, system that is known to possess dimension variability is the double rotor map [7]. This is a four-dimensional invertible map. For our numerical simulations below, we employ the parameter settings of Ref. [13] with $f=8$ (see also Ref. [8]).

B. Systems with homoclinic tangencies

The Hénon map is a paradigm of a system with homoclinic tangencies, which was used to investigate many of the particularities that depend on this fact, such as the creation of a generating partition and the effects of noise [14,2]

$$(x_{n+1},y_{n+1})=(a-x_n^2+by_n,x_n)\quad\text{with } a=1.4,\ b=0.3. \tag{5}$$

The Ikeda map [16]

$$z_{n+1}=1+0.9z_n\exp\left(0.4i-\frac{6i}{1+|z_n|^2}\right),\quad z_n,z_{n+1}\in\mathbf{C} \tag{6}$$

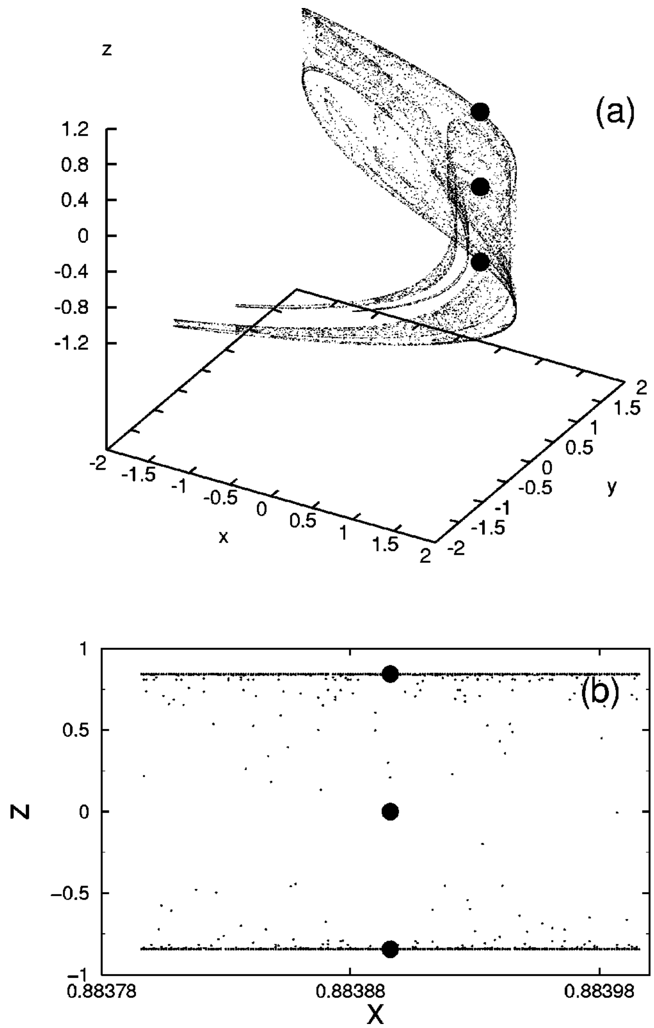

FIG. 3. (a) Attractor of Eq. (7) for $\alpha=3$ in the three-dimensional phase space. (b) Projection of the attractor into the (x,z) plane. The three fixed points are shown by filled circles (●).

possesses homoclinic tangencies as well [2]. Since both maps are two-dimensional and invertible, dimension variability is trivially excluded.

C. System with both types of nonhyperbolicity

Finally, we construct a system that has both types of violation of hyperbolicity, homoclinic tangencies, and dimension variability. Although it seems that this is the general case for hyperchaotic maps and flows, we verify this for our model system. We replace the cat-map dynamics in Eq. (4) by the Hénon map. For additional simplicity, we can relax the invariance condition of the z dynamics in $x-y\to x-y\pm 1$ and hence arrive at the system:

$$x_{n+1}=1-ax_n^2+by_n,$$

$$y_{n+1}=x_n,$$

$$z_{n+1}=\frac{2}{\pi}\arctan\left[\frac{\alpha\pi}{2}\left(z-\frac{x-y}{2}\right)\right]. \qquad (7)$$

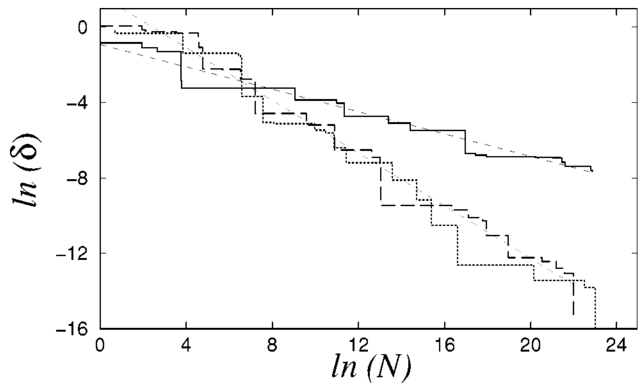

FIG. 4. Minimum distances δ between a chaotic trajectory of length N of Eq. (7) and the fixed points P_1 (long-dashed line), P_2 (solid line), and P_3 (dotted line) on a logarithmic scale. A linear regression gives the following pointwise dimensions: $D_1\approx 1.62$, 2.90, and 1.66 for P_1, P_2, and P_3, respectively.

The Hénon map has two fixed points, one of them, $x_*=y_*=(b-1+\sqrt{(b-1)^2+4a})/2a$, is on the chaotic attractor. The z dynamics, Eq. (7), possesses, for $\alpha<1$, the only fixed point $z_*=0$. At $\alpha=1$ it undergoes a pitchfork bifurcation, so that for $\alpha>1$ the following three fixed points exist: $P_1=(x_*,y_*,-z_*)$, $P_2=(x_*,y_*,0)$, and $P_3=(x_*,y_*,z_*)$, where z_* denotes the nonzero solutions of

$$z_*=\frac{2}{\pi}\arctan\left(\frac{\alpha\pi}{2}z_*\right).$$

All these fixed points possess the following two common eigenvalues: λ_1, $\lambda_2=-x_*\pm\sqrt{x_*^2+0.3}$, with magnitudes larger and less than one, respectively. The third eigenvalue is

$$\lambda_3=\frac{\alpha}{[(\alpha\pi z_*/2)^2+1)]}, \quad \text{for } P_1 \text{ and } P_3, \qquad (8)$$

$$\lambda_3=\alpha, \quad \text{for } P_2.$$

Thus, for $\alpha>1$, we have $|\lambda_3|>1$ for P_2 and $\lambda_3<1$ for P_1 and P_3. That is, the fixed points P_1 and P_3 have one unstable direction while P_2 has two. Figure 3(a) and 3(b) show the chaotic attractor, together with the locations of the three fixed points, in the three-dimensional phase space and in the (x,z) plane, respectively. In order to be confident that the three fixed points are embedded in the chaotic attractor, we compute δ, the smallest distance between a trajectory of length N and the fixed points. If a fixed point P is indeed embedded in the attractor, then the distance decreases algebraically as N is increased, as follows [17]:

$$\delta\sim N^{-1/D_p(P)}, \qquad (9)$$

where $D_p(P)$ is the pointwise dimension of the fixed point. Figure 4 shows the scaling of δ with N for the three fixed points. A linear regression gives $D_p\approx 1.62$, 2.90, and 1.66 for P_1, P_2, and P_3, respectively. The apparent algebraic scaling behavior indicates that the fixed points are embedded in the attractor. Moreover, the values of the pointwise dimension

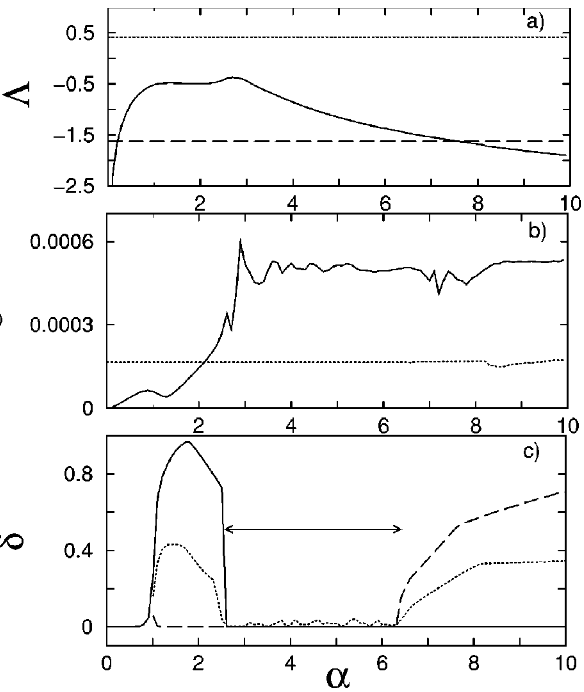

FIG. 5. (a) Lyapunov spectrum of Eq. (7) vs the parameter α. (b) The variances in the second (solid line) and the third (dotted line) Lyapunov exponents. The variances are computed from 20 trajectories of length 5×10^6 each. (c) The minimum distances δ between a chaotic trajectory of length 10^8 with the fixed points P_1 (solid line), P_2 (dotted line), and P_3 (long dashed line).

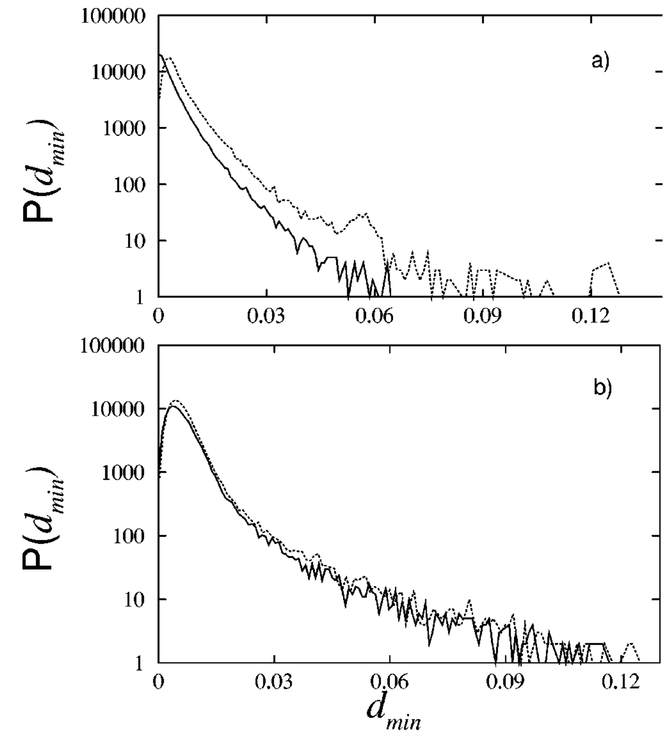

FIG. 6. Distribution of $d_{\min}$ for (a) Eq. (7) for $\alpha=3$, and (b) for Eq. (4) with $\alpha=2$. The solid and dotted lines correspond to noise amplitude of $\epsilon=10^{-4}$ and 4×10^{-3}, respectively. The distributions are obtained by accumulating 10^5 values of $d_{\min}$ from 10^5 points on the noisy attractor and each value of $d_{\min}$ is computed using 10^5 points on the noiseless attractor.

suggest that the attractor is one-dimensionally unstable near P_1 and P_3 and it is two-dimensionally unstable near P_2. Since unstable dimension variability occurs throughout the attractor, generally we expect periodic orbits of higher periods to exhibit a similar behavior.

To better assess the degree of unstable dimension variability, we investigate the fluctuations of the finite-time Lyapunov exponents [5]. Figure 5(a) shows the Lyapunov spectrum of Eq. (7) as a function of α. Apparently, the two exponents from the Hénon map remain constant, and only one Lyapunov exponent, originated from the z equation, varies with α. Figure 5(b) shows the variance of the second (solid line) and the third (dotted line) Lyapunov exponents vs α computed from 20 trajectories of length 5×10^6 each. Figure 5(c) shows the minimum distances between a trajectory of length 10^8 and the three fixed points vs α. In the parameter range: $2.6\lesssim\alpha\lesssim6.3$ where $\delta\approx0$ and, apparently, all three fixed points are embedded in the attractor, there is an appreciable fluctuation of the second Lyapunov exponent. Thus, in this parameter interval, there is a suggestion of persistent unstable dimension variability.

D. Scaling of the Hausdorff distance

We can now study the scaling of the Hausdorff distance with noise. To simulate noise, we add a random variable with a uniform distribution in $[-\epsilon, \epsilon]$ to every dynamical variable before every iteration of the maps. Thus, we expect to observe a wide distribution of $d_{\min}$, the minimum distance between a point on the noisy attractor and a long trajectory in the noiseless attractor, as shown in Figs. 6(a) and 6(b) for Eq. (7) and Eq. (4), respectively.

Figure 7 shows, on a logarithmic scale, the Hausdorff distance d_{Hausd} vs ϵ for all the models introduced before. For each value of ϵ, d_{Hausd} is obtained by utilizing at least 10^7 points representing the noise-free attractor and at least 10^6 points for the noisy trajectory (for some of the three- and four-dimensional systems, up to 10^8 points were used). For all systems, the ϵ dependence is absent if the noise-induced distance is smaller than the distance between two independent finite samples of the noise-free attractor, as explained before. This value is system specific and depends on the dynamical range of the variables, on the dimensionality of the attractor, and on the (ir)regularity of the invariant measure, since it is dominated by the largest interpoint distances in the sparsely populated regions of the attractor.

Apparently, with only unstable dimension variability [Eq. (4) and the double rotor map], outside this plateau, $d_{\text{Hausd}}\approx c\epsilon$ with a factor c bounded by unity. The same holds, as expected, for the hyperbolic case of Eq. (4), when for $\alpha=0.9$ the z direction is contracting everywhere. For Eq. (4) and $\alpha=3$ one observes $c\approx0.01$, whereas for $\alpha=0.9$ $c\approx0.05$. This is consistent with the fact that both the Lyapunov exponent corresponding to the z direction, as also the local expansion rates on the attractor boundary, are much smaller for $\alpha=3$ than for $\alpha=0.9$, despite the presence or lack of dimension variability [the second Lyapunov exponent

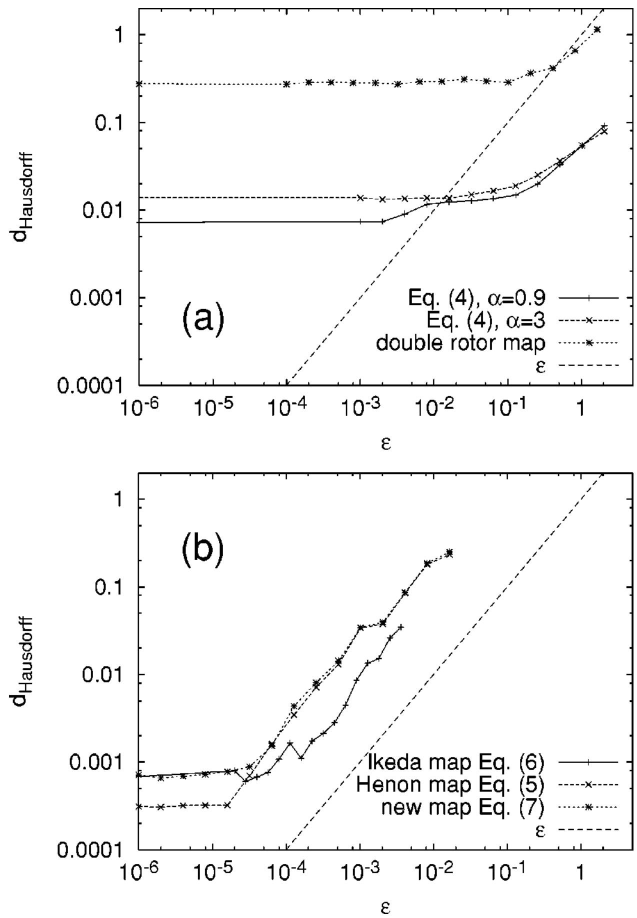

FIG. 7. Hausdorff distances d_{Hausd} vs the noise amplitude ϵ. Panel (a) hyperbolic systems and systems with only dimension variability, Eq. (4) with $\alpha=0.9$ and with $\alpha=3$, and the double rotor map. Panel (b) systems with homoclinic tangencies, the Hénon map, the Ikeda map, and Eq. (7) with $\alpha=3$. The dashed lines indicate $d_{\mathrm{Hausd}}=\epsilon$ in both panels.

assumes the values $\lambda_2=-1.25$ ($\alpha=3$) and $\lambda_2=-0.84$ ($\alpha=0.9$)].

The systems with only homoclinic tangencies, such as the Hénon map and the Ikeda map, differ from these in that homoclinic tangencies give rise to a noise amplification mechanism, which leads to factors c of $c\approx 25$ for the Hénon map and of $c\approx 5-10$ for the Ikeda map. In the latter case, the Hausdorff distance does not fully catch the effect of noise. The attractor is elongated at the homoclinic tangency points with much larger noise amplification factors than 5, but due to the curvature of these attractor arms, they remain in the neighborhood of the noise-free attractor. A side remark is that despite extensive numerical simulations, the scaling regime $d_{\mathrm{hausd}}\propto\epsilon^{1/D_1}$ postulated in Ref. [9] was not reproducible in our experiments.

It is clearly seen that when both unstable dimension variability and tangencies are present (7), the tangency effects are the dominant ones. The Hausdorff distances have to be compared to those of the cat-map driven z dynamics, Eq. (4). In summary, Fig. 7 presents a very surprising result: Nonhyperbolic attractors with homoclinic tangencies are tremendously more sensitive against noise than attractors with only dimension variability. The latter are as robust against noise as hyperbolic sets.

IV. DISCUSSION

An important topic in the study of chaotic systems is whether numerical trajectories generated by computers can be shadowed by true ones [1,6,7]. The focus of this paper is on the behavior of trajectories of nonhyperbolic chaotic systems under experimentally relevant noise levels, rather than just numerical round-off errors. At these noise levels the calculation of shadowing lengths and distance is cumbersome [1]. We investigate how the *entire* noisy attractor is modified compared to the deterministic one. Specifically, we address how the Hausdorff distance between the noisy attractor and the deterministic one behaves when the sources of nonhyperbolicity are unstable dimension variability, homoclinic tangencies, and both.

While nonhyperbolic systems with unstable dimension variability are strongly not shadowable, paradoxically the noisy attractor tends to stay close to the deterministic one in the sense that the Hausdorff distance grows linearly with the noise level with a noise amplification factor smaller than unity. In contrast to that, systems with homoclinic tangencies with much less severe lack of shadowability have noise amplification factors much larger than unity. When a nonhyperbolic system has both unstable dimension variability and tangencies, the effect of tangencies is dominant. These results imply that although unstable dimension variability has a significantly detrimental effect on shadowing, its presence causes the minimal possible effect on the global structure of the attractor under noise.

Similar issues as discussed here have also been addressed in the mathematical literature. For the case of mere round-off errors (i.e., modified deterministic dynamics), the Hausdorff distance has been shown to grow linearly in ϵ as stated by Eq. (3), where ϵ here is related to the magnitude of the perturbation of the system introduced by the discretiazation. To derive these results, essentially the global attractiveness of the attractor has been used, and the internal structure (in particular, whether the attractor is a hyperbolic set or not) is irrelevant for the proof. Hence, the essential prefactor c that is in the focus of our interest is not investigated. Recent extensions reported in Ref. [19] seem to indicate that these results also hold for stochastic perturbations. The conditions to be fulfilled by the dynamics such that the results of Refs. [18], [19] hold are restrictive and examples are yet missing, so that altogether there is still a huge gap in between our essentially numerical and the mathematically rigourous results.

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